J. DIFFERENTIAL GEOMETRY
45 (1997) 78-98

EQUIVARIANT HOLOMORPHIC MORSE INEQUALITIES I: A HEAT KERNEL PROOF

VARGHESE MATHAI & SIYE WU

Abstract

Assume that the circle group acts holomorphically on a compact Kähler manifold with isolated fixed points and that the action can be lifted holomorphically to a holomorphic vector bundle. We use some techniques developed by Bismut and Lebeau to give a heat kernel proof of the equivariant holomorphic Morse inequalities, which, first obtained by Witten using a different argument, produce bounds on the multiplicities of weights occurring in the twisted Dolbeault cohomologies in terms of the data of the fixed points.

1. Introduction

Morse theory gives some topological information of manifolds by means of the critical points of functions. Let h be a Morse function on a compact manifold of real dimension n and suppose that h has isolated critical points only. Let m_k $(0 \leq k \leq n)$ be the k-th Morse number, the number of critical points of Morse index k. The Hopf formula for the gradient vector field says that the alternating sum of m_k is equal to that of the Betti numbers b_k:

$$\sum_{k=0}^{n}(-1)^k m_k = \sum_{k=0}^{n}(-1)^k b_k. \tag{1.1}$$

Received March 15, 1996, and, in revized form, March 24, 1997. The second author was partially supported by NSF grants DMS-93-05578 at Columbia University, DMS-90-22140 at MSRI and by ICTP.

Replacing (-1) by t, we get two polynomials (Morse and Poincaré polynomials, respectively) in t that are equal at $t=-1$, i.e.,

$$\sum_{k=0}^{n} m_k t^k = \sum_{k=0}^{n} b_k t^k + (1+t)q(t) \tag{1.2}$$

for some polynomial $q(t)=\sum_{k=0}^{n} q_k t^k$. The (strong) Morse inequalities assert that $q(t)\geq 0$ in the sense that $q_k \geq 0$ for every $0\leq k\leq n$.

In a celebrated paper [14], Witten showed that the Morse inequalities can be derived by using a deformation

$$d_h = e^{-h} d e^{h} \tag{1.3}$$

of the exterior derivative of the de Rham complex (Ω^*, d). Subsequently Bismut [2] gave a heat kernel proof of the Morse inequalities. Let d^* and d_h^* be the (formal) adjoints of d and d_h (with a choice of Riemannian metrics), respectively, and let

$$\Delta = \{d, d^*\} \quad \text{and} \quad \Delta_h = \{d_h, d_h^*\} \tag{1.4}$$

be the corresponding Laplacians. (We adopt the standard notation of operator (anti-)commutators $\{A,B\}=AB+BA$ and $[A,B]=AB-BA$.) By Hodge theory,

$$\sum_{k=0}^{n} (-1)^k \operatorname{Tr}_{\Omega^k} \exp(-u^2\Delta) = \sum_{k=0}^{n} (-1)^k b_k \tag{1.5}$$

for any $u>0$; this is in fact the starting point of the heat kernel proof of the index theorem. Similarly, after replacing (-1) by t, we obtain

$$\sum_{k=0}^{n} t^k \operatorname{Tr}_{\Omega^k} \exp(-u^2\Delta) = \sum_{k=0}^{n} b_k t^k + (1+t) q_u(t). \tag{1.6}$$

It is a straightforward consequence of Hodge theory that the polynomial $q_u(t)\geq 0$. (See for example [2, Theorem 1.3]. A slightly different method is used to show the equivariant version in Lemma 4.1 below.) Since (Ω^*, d_h) defines the same cohomology groups as (Ω^*, d), we can replace the heat kernels in (1.6) by those associated to the deformed Laplacian Δ_h. It turns out that

$$\lim_{T\to+\infty} \lim_{u\to+0} \operatorname{Tr}_{\Omega^k} \exp(-u^2 \Delta_{Th/u^2}) = m_k \quad (0\leq k\leq n); \tag{1.7}$$

the (strong) Morse inequalities (1.2) then follow. The heart of the proof is that as $u \to 0$ the heat kernel is localized near the critical points of h, around which the operator consists of n copies of (supersymmetric) harmonic oscillators whose heat kernels are given by Mehler's formula.

Witten [15] also introduced a holomorphic analog of [14]. Let M be a compact Kähler manifold of complex dimension n and let E be a holomorphic vector bundle over M. Let $H^k(M, \mathcal{O}(E))$ be the cohomology groups with coefficients in the sheaf of holomorphic sections of E, calculated from the twisted Dolbeault complex $(\Omega^{0,*}(M,E), \bar\partial_E)$. Suppose that the circle group S^1 acts holomorphically and effectively on M preserving the Kähler structure and that the action can be lifted holomorphically to E. We denote the actions of $e^{\sqrt{-1}\theta} \in S^1$ on M and E by $g(\theta)$ and $\tilde g(\theta)$, respectively. Then $e^{\sqrt{-1}\theta}$ also acts on the space of sections by sending a section s to $\tilde g(\theta) \circ s \circ g^{-1}(\theta)$. The induced action on $\Omega^{0,*}(M,E)$ commutes with the operator $\bar\partial_E$. Thus we obtain representations $\bar g(\theta)$ of $e^{\sqrt{-1}\theta} \in S^1$ on $H^k(M, \mathcal{O}(E))$; the multiplicities of weights of S^1 in each cohomology group will be the subject of our investigation. The S^1-action on (M, ω) is clearly symplectic: let V be the vector field on M that generates the S^1-action; then $L_V\omega = 0$. If the fixed-point set F of S^1 on M is non-empty, then the S^1-action is Hamiltonian [9], i.e., there is a moment map $h\colon M \to \mathbb{R}$ such that $i_V\omega = dh$. We further assume that F contains isolated points only. It is well known that all the Morse indices are even and hence by the lacunary principle, h is a perfect Morse function: $m_{2k-1} = b_{2k-1} (= 0)$, and $m_{2k} = b_{2k}$ $(0 \le k \le n)$. However a refined statement is possible because of the complex structure. For each $p \in F$, S^1 acts on T_pM by the isotropic representation; let $\lambda_1^p, \cdots, \lambda_n^p \in \mathbb{Z}\backslash\{0\}$ be the weights. We define the *polarizing index* n^p of the fixed point $p \in F$ as the number of weights $\lambda_k^p < 0$; the Morse index of h at p is then $2(n - n_p)$. (We need to explain our convention in a simple (but non-compact) example $M = \mathbb{C}$, $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\bar z$, with an S^1-action of weight $\lambda \in \mathbb{Z}\backslash\{0\}$. Since $V = \sqrt{-1}\lambda(z\frac{\partial}{\partial z} - \bar z\frac{\partial}{\partial \bar z})$, we have $h = -\frac{1}{2}\lambda|z|^2$. Also, the weight of the S^1-action on the function z^k (a section of the trivial bundle) is $-k\lambda$ $(k \in \mathbb{Z}, k \ge 0)$; this leads to a sign convention different from [15] in the main result.) Furthermore, S^1 acts on the fiber E_p over $p \in F$. It is useful to recall a notation in [15]. If the group S^1 has a representation on a finite dimensional complex vector space W, let $W(\theta)$ $(\theta \in \mathbb{R})$ be its character. For example, we denote $E_p(\theta) = \mathrm{tr}_{E_p}\tilde g(\theta)$ and $H^k(\theta) = \mathrm{tr}_{H^k(M,\mathcal{O}(E))}\bar g(\theta)$. The analog of the Hopf formula (1.1) is the fixed-point formula of Atiyah and Bott

[1], which we write as an equality of alternating sums [14]:

$$\begin{aligned}\sum_{p\in F}(-1)^{n_p}E_p(\theta)\prod_{\lambda_k^p>0}\frac{1}{1-e^{-\sqrt{-1}\lambda_k^p\theta}}\prod_{\lambda_k^p<0}\frac{e^{-\sqrt{-1}|\lambda_k^p|\theta}}{1-e^{-\sqrt{-1}|\lambda_k^p|\theta}}\\ =\sum_{k=0}^{n}(-1)^kH^k(\theta).\end{aligned}\tag{1.8}$$

It turns out that if (-1) is replaced by t, the analog of strong Morse inequalities such as (1.2) holds. We need

Definition 1.1. Let $q(\theta)=\sum_{m\in\mathbb{Z}}q_me^{\sqrt{-1}m\theta}\in\mathbb{R}((e^{\sqrt{-1}\theta}))$ be a formal character of S^1. Then we say $q(\theta)\geq 0$ if $q_m\geq 0$ for all $m\in\mathbb{Z}$. Let $Q(\theta,t)=\sum_{k=0}^{n}q_k(\theta)t^k\in\mathbb{R}((e^{\sqrt{-1}\theta}))[t]$ be a polynomial of degree n with coefficients in $\mathbb{R}((e^{\sqrt{-1}\theta}))$. Then we say $Q(\theta,t)\geq 0$ if $q_k(\theta)\geq 0$ for all k.

For example, the characters $E_p(\theta)$ and $H^k(\theta)$ are elements of $\mathbb{R}((e^{\sqrt{-1}\theta}))$. Another important type of elements in $\mathbb{R}((e^{\sqrt{-1}\theta}))$ is given by the series

$$\frac{e^{\sqrt{-1}\eta\theta}}{1-c\,e^{\sqrt{-1}\xi\theta}}\overset{\text{def.}}{=}\sum_{k=0}c^ke^{\sqrt{-1}(k\xi+\eta)\theta}\qquad(c\in\mathbb{R},\ \ \xi,\eta\in\mathbb{Z}).\tag{1.9}$$

We emphasize here that in (1.9) the left-hand side is a notation for the formal series on the right-hand side. (1.8) can now be regarded as an equality in $\mathbb{R}((e^{\sqrt{-1}\theta}))$.

Using a holomorphic version of supersymmetric quantum mechanics, Witten [15] derived the following.

Theorem 1.2. *Suppose that M is a compact Kähler manifold on which S^1 acts holomorphically preserving the Kähler form and with non-empty and discrete fixed-point set, and suppose that the S^1-action can be lifted holomorphically to a holomorphic vector bundle E over M. Then we have the strong equivariant holomorphic Morse inequalities:*

$$\begin{aligned}\sum_{p\in F}t^{n_p}E_p(\theta)\prod_{\lambda_k^p>0}\frac{1}{1-e^{-\sqrt{-1}\lambda_k^p\theta}}\prod_{\lambda_k^p<0}\frac{e^{-\sqrt{-1}|\lambda_k^p|\theta}}{1-e^{-\sqrt{-1}|\lambda_k^p|\theta}}\\ =\sum_{k=0}^{n}t^kH^k(\theta)+(1+t)Q^+(\theta,t),\end{aligned}\tag{1.10}$$

$$\begin{aligned}(1.11)\qquad \sum_{p\in F} t^{n-n_p} E_p(\theta) \prod_{\lambda_k^p>0} \frac{e^{\sqrt{-1}\lambda_k^p\theta}}{1-e^{\sqrt{-1}\lambda_k^p\theta}} \prod_{\lambda_k^p<0} \frac{1}{1-e^{\sqrt{-1}|\lambda_k^p|\theta}} \\ = \sum_{k=0}^{n} t^k H^k(\theta) + (1+t) Q^-(\theta,t),\end{aligned}$$

where $Q^{\pm}(\theta,t) \geq 0$ *in* $\mathbb{R}((e^{\sqrt{-1}\theta}))$.

As is standard in Morse theory, the strong inequalities (1.10) and (1.11) imply the corresponding weak inequalities, and the fixed-point formula of Atiyah and Bott can be recovered by setting $t=-1$ in either (1.10) or (1.11). Furthermore, we obtain (1.10) from (1.11) after reversing the S^1-action and replacing θ by $-\theta$. The whole paper is devoted to a heat kernel proof of (1.11).

The cohomology groups $H^k(M,\mathcal{O}(E))$ as representation spaces of S^1 depend only on the (S^1-invariant) holomorphic structure on E. We can choose an S^1-invariant Hermitian form on E and let $d_E = \partial_E + \bar\partial_E$ be the unique compatible holomorphic connection. To simplify the notation, we drop the subscript E but keep in mind that

$$(1.12)\qquad \partial^2 = \bar\partial^2 = 0 \qquad \text{and} \qquad d^2 = \{\partial, \bar\partial\} = \Omega \wedge \cdot\,,$$

where the curvature Ω is a $(1,1)$-form on M with values in $\mathrm{End}(E)$. Let d^*, ∂^*, $\bar\partial^*$ be the (formal) adjoints of d, ∂, $\bar\partial$, respectively and let

$$(1.13)\qquad \Delta = \{d, d^*\}, \quad \Box = \{\partial, \partial^*\}, \quad \overline{\Box} = \{\bar\partial, \bar\partial^*\}$$

be the corresponding Laplacians. Following Witten [15], we deform the $\bar\partial$-operator and its Laplacian by

$$(1.14)\qquad \bar\partial_h = e^{-h}\bar\partial e^h, \quad \bar\partial_h^* = e^h \bar\partial^* e^{-h}, \quad \overline{\Box}_h = \{\bar\partial_h, \bar\partial_h^*\}.$$

The analog of (1.6) holds, where b_k should be replaced by $\dim H^k(M,\mathcal{O}(E))$, $0 \leq k \leq n$. Contrary to the treatment of ordinary Morse theory in [14], [2], the limit of $\mathrm{Tr}_{\Omega^{0,k}(M,E)} \exp(-u^2 \overline{\Box}_{Th/u^2})$ as $u \to 0$ does not exist. To see this, we observe that [15] (see also formulas (2.16) and (2.22) below) up to a (bounded) 0-th order operator, $u^2\overline{\Box}_{Th/u^2}$ is equal to $\frac{1}{2}u^2\Delta_{Th/u^2} - \sqrt{-1}TL_V$. Here L_V is, up to a sign, the infinitesimal action of the circle group S^1; it is an (unbounded) first order differential operator. Simple arguments such as finite propagation speed show that as $u \to 0$, localization does not take place for the smooth kernel of $u^2\overline{\Box}_{Th/u^2}$. From the physics point of view, the operator $u^2\overline{\Box}_{Th/u^2}$ near a critical point of h is the Hamiltonian operator of a (supersymmetric) charged particle in a uniform magnetic field. Therefore the wave functions, and hence the kernel, do not localize to any preferred point in the strong field limit. However since L_V commutes with $u^2\overline{\Box}_{Th/u^2}$, we can restrict the latter to the eigenspaces of the former, on each of which L_V is a constant. This leads naturally equivariant Morse-type inequalities [15]. In the heat kernel approach of this paper, we absorb the operator $\sqrt{-1}TL_V$ into θL_V that appears in the S^1-equivariant kernel. The remaining operator $u^2\overline{\Box}_{Th/u^2} + \sqrt{-1}TL_V$, which is equal to $\frac{1}{2}u^2\Delta_{Th/u^2}$ up to a 0-th order term, does have a localized heat kernel as $u \to 0$. Therefore the problem is reduced to calculating the S^1-equivariant heat kernel of harmonic oscillators, which can be obtained from Mehler's formula.

The inequalities due to Demailly [8] have also been referred to in the literature as holomorphic Morse inequalities. The important difference with our case is that Demailly's inequalities do not require a group action and are asymptotic inequalities, as the tensor power of a holomorphic line bundle gets large, whereas the inequalities which we consider are for a fixed holomorphic vector bundle with a holomorphic S^1-action, and are not merely asymptotic.

In Section 2, we study various deformations of the Laplacians on Kähler manifolds. In particular, the operator $\overline{\Box}_h$ is calculated explicitly. We also compare two other deformations $\overline{\Box}_v$ and $\overline{\Box}_{\sqrt{-1}v}$, which are used in studying complex immersions [5] and holomorphic equivariant cohomology groups [12]. Roughly speaking, the three operators $\frac{1}{2}\Delta_h$, $\overline{\Box}_v$ and $\overline{\Box}_{\sqrt{-1}v}$ form a triplet of a certain $SU(2)$ group. In Section 3, we use the technique of [5] to show that as $u \to 0$, the smooth heat kernel associated to the operator $\exp(-u^2\overline{\Box}_{Th/u^2} - \sqrt{-1}TL_V)$ ($u > 0$, $T > 0$) is localized near the fixed-point set F, and when F is discrete, the equivariant heat kernel can be approximated by using the operators with coefficients frozen at the fixed points. The result of the previous section is used to relate by a unitary conjugation the operator $-u^2\overline{\Box}_{Th/u^2} - \sqrt{-1}TL_V$ to $-u^2\overline{\Box}_{Tv/u^2}$ that appears in [5] (but restricted to a certain subspace) plus a 0-th order operator $-\sqrt{-1}Tr_V$ (as $u \to 0$) whose action does not depend on the degree of differential forms. This has enabled us to follow the analysis of [5] closely, though a more direct approach without using the conjugation also seems possible. In Section 4, we calculate the equivariant heat kernel of the linearized problem using Mehler's formula and then deduce the (strong) equivariant holomorphic

Morse inequalities (1.11) by taking the limit $T \to +\infty$. Unlike the argument using small eigenvalues [15], the 0-th order operator r_V plays a crucial role in the heat kernel calculation.

In a separate paper [16], equivariant holomorphic Morse inequalities with torus and non-Abelian group actions are established and are applied to toric and flag manifolds. It is also shown that the Kähler assumption is necessary for the strong inequalities [16]. The situations with non-isolated fixed points are left for further investigation.

2. Deformed Laplacians on Kähler manifolds

Recall that E is a holomorphic Hermitian vector bundle over a compact Kähler manifold (M,ω). (The Hermitian structure is introduced in the proof but not needed in the statement of Theorem 1.2.) Let $\Lambda_+ = \omega\wedge\cdot$ be the exterior multiplication of ω on $\Omega^{*,*}(M,E)$ and $\Lambda_- = \Lambda_+^*$, its adjoint. Then $\Lambda_3 = \frac{1}{2}[\Lambda_+,\Lambda_-]$ preserves the bi-grading of $\Omega^{*,*}(M,E)$. In fact, the action of Λ_3 on $\Omega^{p,q}(M,E)$ is $\frac{1}{2}(p+q-n)$, hence $[\Lambda_3,\Lambda_\pm] = \pm\Lambda_\pm$. Set $\Lambda_1 = \frac{1}{2}(\Lambda_+ + \Lambda_-)$ and $\Lambda_2 = -\frac{\sqrt{-1}}{2}(\Lambda_+ - \Lambda_-)$. Then Λ_a $(a=1,2,3)$ satisfy the standard $\mathfrak{su}(2)$ commutation relations

$$[\Lambda_a,\Lambda_b] = \sqrt{-1}\epsilon_{abc}\Lambda_c. \tag{2.1}$$

(See for example [10].) So there is a unitary representation of $SU(2)$ on $\Omega^{*,*}(M,E)$; let $S_a(\alpha) = e^{\sqrt{-1}\alpha\Lambda_a}$ be the corresponding group elements. We now introduce a slightly more generalized setup.

Definition 2.1. Let $\sigma\in\Omega^{1,1}(M,E)$ be a real-valued $(1,1)$-form. Set $\Lambda_+(\sigma) = \sigma\wedge\cdot$, $\Lambda_-(\sigma) = \Lambda_+^*$, $\Lambda_1(\sigma) = \frac{1}{2}(\Lambda_+(\sigma)+\Lambda_-(\sigma))$, $\Lambda_2(\sigma) = -\frac{\sqrt{-1}}{2}(\Lambda_+(\sigma)-\Lambda_-(\sigma))$ and $\Lambda_3(\sigma) = \frac{1}{2}[\Lambda_+,\Lambda_-(\sigma)](=-\frac{1}{2}[\Lambda_-,\Lambda_+(\sigma)])$.

Remark 2.2. In computations, it is sometimes convenient to introduce local complex coordinates $\{z^k, k=1,\cdots,n\}$ on M. The Kähler form $\omega = \omega_{k\bar{l}}dz^k\wedge d\bar{z}^{\bar{l}}$ is related to the metric $g = g_{k\bar{l}}dz^k\otimes d\bar{z}^{\bar{l}}$ by $\omega_{k\bar{l}} = \sqrt{-1}g_{k\bar{l}} = -\omega_{\bar{l}k}$. Let e^k, $e^{\bar{l}}$ be the multiplications by dz^k, $d\bar{z}^{\bar{l}}$, and i_k, $i_{\bar{l}}$, the contractions by $\frac{\partial}{\partial z^k}$, $\frac{\partial}{\partial \bar{z}^{\bar{l}}}$, respectively. Clearly they satisfy the following anti-commutation relations: $\{e^k,i_l\} = \delta^k_l$, $\{e^{\bar{k}},i_{\bar{l}}\} = \delta^{\bar{k}}_{\bar{l}}$ and others $=0$. If a $(1,1)$-form $\sigma = \sigma_{k\bar{l}}dz^k\wedge d\bar{z}^{\bar{l}}$ is real-valued, then all the $\sigma_{k\bar{l}}$'s are purely imaginary. Setting $i^k = g^{k\bar{l}}i_{\bar{l}}$ and $i^{\bar{l}} = g^{k\bar{l}}i_k$, we have $\Lambda_+(\sigma) = \sigma_{k\bar{l}}e^ke^{\bar{l}}$, $\Lambda_-(\sigma) = -\sigma_{k\bar{l}}i^ki^{\bar{l}}$, $\Lambda_1(\sigma) = \frac{1}{2}\sigma_{k\bar{l}}(e^ke^{\bar{l}} - i^ki^{\bar{l}})$, $\Lambda_2(\sigma) = -\frac{\sqrt{-1}}{2}\sigma_{k\bar{l}}(e^ke^{\bar{l}} + i^ki^{\bar{l}})$, and $\Lambda_3(\sigma) = -\frac{\sqrt{-1}}{4}\sigma_{k\bar{l}}([e^k,i^{\bar{l}}]+[e^{\bar{l}},i^k])$.

Lemma 2.3.

$$S_1(\alpha)^{-1}\Lambda_3(\sigma)S_1(\alpha) = \cos\alpha\,\Lambda_3(\sigma) - \sin\alpha\,\Lambda_2(\sigma), \tag{2.2}$$
$$S_2(\alpha)^{-1}\Lambda_3(\sigma)S_2(\alpha) = \cos\alpha\,\Lambda_3(\sigma) + \sin\alpha\,\Lambda_1(\sigma), \tag{2.3}$$
$$S_3(\alpha)^{-1}\Lambda_3(\sigma)S_3(\alpha) = \Lambda_3(\sigma). \tag{2.4}$$

Proof. A straightforward calculation using the above anti-commutation relations shows that $[\Lambda_a, \Lambda_b(\sigma)] = \sqrt{-1}\epsilon_{abc}\Lambda_c(\sigma)$. This means that $\{\Lambda_a(\sigma)\}$ is an $SU(2)$ triplet. Hence the result. q.e.d.

It is clear that the Hodge relations (see for example [10])

$$[\Lambda_-, \partial] = \sqrt{-1}\bar{\partial}^*, \qquad [\Lambda_-, \bar{\partial}] = -\sqrt{-1}\partial^* \tag{2.5}$$
$$[\Lambda_+, \partial^*] = \sqrt{-1}\bar{\partial}, \qquad [\Lambda_+, \bar{\partial}^*] = -\sqrt{-1}\partial \tag{2.6}$$

still hold after coupling to the vector bundle E. Moreover, we have the Bochner-Kodaira-Nakano identities

$$\Delta = \Box + \overline{\Box}, \qquad \overline{\Box} - \Box = 2\Lambda_3(\sqrt{-1}\Omega), \tag{2.7}$$

which are consequences of (2.5), (2.6) and the graded Jacobi identities. (Since E is a holomorphic Hermitian bundle, $\sqrt{-1}\Omega$ is a $(1,1)$-form valued in the subset of $\mathrm{End}(E)$ which consists of self-adjoint endomorphisms.) These results have been generalized to non-Kähler situations in [7]. When E is a flat bundle, we recover the usual relation $\Box = \overline{\Box} = \frac{1}{2}\Delta$.

Lemma 2.4.

$$\begin{aligned} S_1(\alpha)^{-1}\overline{\Box}S_1(\alpha) =& \overline{\Box} - (1-\cos\alpha)\Lambda_3(\sqrt{-1}\Omega) \\ & - \sin\alpha\,\Lambda_2(\sqrt{-1}\Omega), \end{aligned} \tag{2.8}$$
$$\begin{aligned} S_2(\alpha)^{-1}\overline{\Box}S_2(\alpha) =& \overline{\Box} - (1-\cos\alpha)\Lambda_3(\sqrt{-1}\Omega) \\ & + \sin\alpha\,\Lambda_1(\sqrt{-1}\Omega), \end{aligned} \tag{2.9}$$
$$S_3(\alpha)^{-1}\overline{\Box}S_3(\alpha) = \overline{\Box}. \tag{2.10}$$

Proof. From (2.5) and (2.6) we deduce that

$$S_1(\alpha)^{-1}\bar{\partial}S_1(\alpha) = \cos\frac{\alpha}{2}\bar{\partial} - \sin\frac{\alpha}{2}\partial^*.$$

Therefore

$$\begin{aligned} & S_1(\alpha)^{-1}\overline{\Box}S_1(\alpha) \\ & = \{\cos\frac{\alpha}{2}\,\bar\partial - \sin\frac{\alpha}{2}\,\partial^*, \cos\frac{\alpha}{2}\,\bar\partial^* - \sin\frac{\alpha}{2}\,\partial\} \\ & = \cos^2\frac{\alpha}{2}\overline{\Box} + \sin^2\frac{\alpha}{2}\,\Box - \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(\{\partial,\bar\partial\} + \{\partial^*,\bar\partial^*\}) \\ & = \overline{\Box} - (1-\cos\alpha)\Lambda_3(\sqrt{-1}\Omega) - \sin\alpha\,\Lambda_2(\sqrt{-1}\Omega). \end{aligned} \tag{2.11}$$

The second formula follows in the same fashion from $S_2(\alpha)^{-1}\bar\partial S_2(\alpha) = \cos\frac{\alpha}{2}\bar\partial - \sqrt{-1}\sin\frac{\alpha}{2}\partial^*$. The last one holds because $\overline{\Box}$ preserves the bigrading. q.e.d.

We now equip M with a holomorphic S^1-action which preserves the Kähler structure, hence both the complex structure J and the Riemannian metric g. The holomorphic condition $L_V J = 0$ and the Killing equation $L_V g = 0$ read, in components,

$$V_{k;l} = V_{\bar k;\bar l} = 0 \qquad \text{and} \qquad V_{k,\bar l} + V_{\bar l,k} = 0, \tag{2.12}$$

respectively. As explained in Section 1, we assume that the S^1 fixed-point set F is non-empty. In this case, there is a moment map $h\colon M \to \mathbb{R}$ satisfying $i_V\omega = dh$, or $h_{,k} = -\sqrt{-1}V_k$ and $h_{,\bar k} = \sqrt{-1}V_{\bar k}$. The equations in (2.12) are equivalent to

$$h_{,k;l} = h_{,\bar k;\bar l} = 0 \qquad \text{and} \qquad h_{,k,\bar l} = h_{,\bar l,k}. \tag{2.13}$$

(The second part is of course the symmetry of the Hessian.) Also notice the real-valued $(1,1)$-form

$$dJdh = di_V g = -2\sqrt{-1}h_{,k,\bar l}dz^k \wedge d\bar z^{\bar l}. \tag{2.14}$$

We further assume that the S^1-action can be lifted holomorphically to the bundle E. We can choose an S^1-invariant Hermitian form on E. Then the connection $d = d_E$ is also S^1-invariant. The group element $e^{\sqrt{-1}\theta} \in S^1$ acts on a section s by $s \mapsto \tilde g(\theta) \circ s \circ g^{-1}(\theta)$. Let L_V be the Lie derivative of E-valued forms along V. (The fibers of E over different points on the integral curve of V are related by the lifted S^1-action.) Then $-L_V$ is the infinitesimal generator of the S^1-action on $\Omega^{*,*}(M,E)$. Let $\overline{L}_V = \{i_V, d\}$, where $d = d_E$. Then the operator

$$r_V = \overline{L}_V - L_V \tag{2.15}$$

is an element of $\Gamma(M, \mathrm{End}(E))$. Over the fiber of a fixed point $p \in F$, $r_V(p)$ is simply the representation of $\mathrm{Lie}(S^1)$ on E_p; this is independent of the choice of the connection on E.

Remark 2.5. Recall that $\overline{\Box}_h$ is the Laplacian of Witten's deformed Dolbeault operator $\bar\partial_h$.

1. $\overline{\Box}_h$ commutes with the S^1-action: Since the connection, the complex structure, and the moment map h are all S^1-invariant, we get $[L_V, d] = 0$, $[L_V, \bar\partial] = 0$ and $[L_V, \bar\partial_h] = 0$. Taking the adjoint gives $[L_V, \bar\partial_h^*] = 0$ and $[L_V, \overline{\Box}_h] = 0$.

2. $\overline{\Box}_h$ also commutes with a $U(1)$ subgroup of $SU(2)$: Since $\overline{\Box}_h$ preserves the bi-grading, $[\Lambda_3, \overline{\Box}_h] = 0$, hence $S_3(\alpha)^{-1}\overline{\Box}_h S_3(\alpha) = \overline{\Box}_h$.

Proposition 2.6.

$$\overline{\Box}_h = \overline{\Box} + \tfrac{1}{2}|dh|^2 - \Lambda_3(dJdh) - \sqrt{-1}r_V - \sqrt{-1}L_V. \tag{2.16}$$

Proof. Let D_k, $D_{\bar l}$ be the covariant derivative along $\frac{\partial}{\partial z^k}$, $\frac{\partial}{\partial \bar z^l}$, respectively. Then $\bar\partial = e^{\bar l} D_{\bar l}$, $\bar\partial^* = -i^k D_k$ and $\bar\partial_h = \bar\partial + e^{\bar l} h_{,\bar l}$, $\bar\partial_h^* = \bar\partial^* + i^k h_{,k}$. So

$$\begin{aligned}\overline{\Box}_h =&\{\bar\partial, \bar\partial^*\} + \{e^{\bar l}, i^k\} h_{,\bar l} h_{,k} + \{\bar\partial, i^k h_{,k}\} + \{\bar\partial^*, e^{\bar l} h_{,\bar l}\}\\ =&\overline{\Box} + g^{k\bar l} h_{,k} h_{,\bar l} + (h_{,k,\bar l}\, e^{\bar l} i^k + h_{,k} D^k) - (h_{,\bar l,k} i^k e^{\bar l} + h_{,\bar l} D^{\bar l})\\ =&\overline{\Box} + \tfrac{1}{2}|dh|^2 - \Lambda_3(dJdh) + h_{,k,\bar l}(e^{\bar l} i^k - e^k i^{\bar l})\\ &- \sqrt{-1}(V_k D^k + V_{\bar l} D^{\bar l})\\ =&\overline{\Box} + \tfrac{1}{2}|dh|^2 - \Lambda_3(dJdh) - \sqrt{-1}\{\partial + \bar\partial, V_k i^k + V_{\bar l} i^{\bar l}\}.\end{aligned} \tag{2.17}$$

The last anti-commutator is $\overline{L}_V = r_V + L_V$. q.e.d.

We also define two different deformations. Let $v = V^{1,0}$ be the holomorphic component of V. Set

$$\bar\partial_v = \bar\partial + i_v, \qquad \overline{\Box}_v = \{\bar\partial_v, \bar\partial_v^*\} \tag{2.18}$$

and

$$\bar\partial_{\sqrt{-1}v} = \partial + \sqrt{-1} i_v, \qquad \overline{\Box}_{\sqrt{-1}v} = \{\bar\partial_{\sqrt{-1}v}, \bar\partial^*_{\sqrt{-1}v}\}. \tag{2.19}$$

Then straightforward calculations similar to what leads to (2.16) yield

$$\overline{\Box}_v = \overline{\Box} + \tfrac{1}{2}|dh|^2 + \Lambda_1(dJdh) \tag{2.20}$$

and

$$\overline{\Box}_{\sqrt{-1}v} = \overline{\Box} + \tfrac{1}{2}|dh|^2 + \Lambda_2(dJdh). \tag{2.21}$$

It is also interesting to compare the deformation Δ_h in [14] of the usual Laplacian (coupled to the bundle E). Using (2.13) again, we get

$$\tfrac{1}{2}\Delta_h = \tfrac{1}{2}\Delta + \tfrac{1}{2}|dh|^2 - \Lambda_3(dJdh). \tag{2.22}$$

When the bundle E is flat, the only difference of $\overline{\Box}_v$, $\overline{\Box}_{\sqrt{-1}v}$ and $\frac{1}{2}\Delta_h$ are in the terms $\Lambda_a(dJdh)$ $(a = 1, 2, 3)$, and the $\frac{\pi}{2}$ rotations in $SU(2)$ interchanges the three operators $\overline{\Box}_v$, $\overline{\Box}_{\sqrt{-1}v}$ and $\frac{1}{2}\Delta_h$.

Finally we come to the relation of $\overline{\Box}_h$ and $\overline{\Box}_v$.

Proposition 2.7.

$$\begin{aligned} &S_2(-\tfrac{\pi}{2})^{-1}\overline{\Box}_h S_2(-\tfrac{\pi}{2}) \\ &\qquad = \overline{\Box}_v - \Lambda_3(\sqrt{-1}\Omega) - \Lambda_1(\sqrt{-1}\Omega) - \sqrt{-1}r_V - \sqrt{-1}L_V. \end{aligned} \tag{2.23}$$

Proof. By means of (2.16), (2.9) and (2.3), we obtain

$$\begin{aligned} S_2(\alpha)^{-1}\overline{\Box}_h S_2(\alpha) = \overline{\Box} &- (1-\cos\alpha)\Lambda_3(\sqrt{-1}\Omega) \\ &+ \sin\alpha\,\Lambda_1(\sqrt{-1}\Omega) + \tfrac{1}{2}|dh|^2 \\ &- \cos\alpha\,\Lambda_3(dJdh) - \sin\alpha\,\Lambda_1(dJdh) \\ &- \sqrt{-1}r_V - \sqrt{-1}L_V. \end{aligned} \tag{2.24}$$

Set $\alpha = -\frac{\pi}{2}$. q.e.d.

3. Localization to the fixed-point set

Definition 3.1. For $u > 0$, $T \geq 0$, let $P_{u,T}(x, x')$, $x, x' \in M$, be the smooth kernel associated to the operator $\exp(-u^2\overline{\Box}_{Th/u} - \sqrt{-1}uTL_V)$ calculated with respect to the Riemannian volume element dv_M of M.

So for $x \in M$, $P_{u,T}(x, x) \in \mathrm{End}(\Omega^{0,*}(M, E))|_x$. Moreover,

$$e^{-\theta L_V} P_{u,T}(g^{-1}(\theta)x, x) \in \mathrm{End}(\Omega^{0,*}(M, E))|_x.$$

Proposition 3.2. *Take $\alpha > 0$, $T > 0$. There exist $c > 0$, $C > 0$ such that for all $x \in M$ with $d(x, F) \geq \alpha$, $e^{\sqrt{-1}\theta} \in S^1$, and all $u \in (0, 1]$, we have*

$$|P_{u,T/u}(g^{-1}(\theta)x, x)| \leq c\,e^{-C/u^2}. \tag{3.1}$$

Proof. We use the techniques (and the notation) of [5]. Consider $i\colon F \to M$ as an embedding of compact complex manifolds. Let $\eta = i^*E$ and $\xi_k = \Lambda^k T^{*(1,0)}M \otimes E$ $(k = 0, \cdots, n)$. Then

$$(\xi, i_v)\colon 0 \to \xi_n \to \xi_{n-1} \to \cdots \to \xi_0 \tag{3.2}$$

is a holomorphic chain complex of vector bundles on M. Since F is discrete, (3.2), together with the restriction map $\xi_0|_F \to \eta$, is a resolution of the sheaf $i_*\mathcal{O}_F(\eta)$. The elliptic operator considered in [5] is

$$u^2\overline{\square}_{Tv/u} = (uD^M + T\hat{V})^2 = u^2(D^M)^2 + uT\{D^M, \hat{V}\} + T^2\hat{V}^2 \tag{3.3}$$

acting on $\Omega^{*,0}(M)\hat{\otimes}\Omega^{0,*}(M,E) = \Omega^{*,*}(M,E)$, where $D^M = \bar{\partial}_v + \bar{\partial}_v^*$, $\hat{V} = i_v + i_v^*$. It is particularly important that the operator $\{D^M, \hat{V}\}$ is of order zero, hence $uT\{D^M, \hat{V}\}$ is uniformly bounded for $u \in (0,1]$, $T \in [0, 1/u]$. We now extend the domain of our operator $\overline{\square}_h$ from (the L^2-completion of) $\Omega^{0,*}(M,E)$ to (that of) $\Omega^{*,*}(M,E)$. Since the operator preserves the bi-grading of $\Omega^{*,*}(M,E)$, it suffices to prove (3.1) for the heat kernel with the extended domain. Using Proposition 2.7, we obtain

$$S_2(-\tfrac{\pi}{2})^{-1}(u^2\overline{\square}_{Th/u} + \sqrt{-1}uTL_V)S_2(-\tfrac{\pi}{2}) = u^2\overline{\square}_{Tv/u} - r_{u,T}. \tag{3.4}$$

Here $r_{u,T} = u\Lambda_3(\sqrt{-1}\Omega) + u\Lambda_1(\sqrt{-1}\Omega) + uT\sqrt{-1}r_V$ is also uniformly bounded for $u \in (0,1]$, $T \in [0, 1/u]$. The operator on the right-hand side of (3.4) has the same heat kernel $P_{u,T}$ up to a conjugation by $S_2(-\frac{\pi}{2})$. Therefore the proof of [5, Proposition 11.10] implies that there exist a sufficiently small $b > 0$ (determined by the injectivity radius of M), and $c_1 > 0$, $C_1 > 0$ such that for all $x_0 \in M$, $u \in (0,1]$, $T \in [0, 1/u]$, $x \in B(x_0, b/2)$, we have

$$|(P_{u,T} - P^{x_0}_{u,T})(x,x)| \le c_1 e^{-C_1/u^2}. \tag{3.5}$$

Here $P^{x_0}_{u,T}$ is the smooth heat kernel of the same operator with Dirichlet conditions on $\partial B^M(x_0, b)$. Hence

$$|(P_{u,T/u} - P^{x}_{u,T/u})(x,x)| \le c_1 e^{-C_1/u^2} \tag{3.6}$$

for all $x \in M$ and all $0 \le T \le 1$. The condition $T \le 1$ can be lifted by a scaling argument. So for any $T > 0$, there exist $c_1, C_1 > 0$ such that (3.6) holds. Since $\hat{V}$ is invertible on $M \backslash F$, by the proof of [5, Proposition 12.1], for any $\alpha > 0$, $T > 0$ there exist $c_2, C_2, C_2' > 0$ such that

$$|P^x_{u,T/u}(x,x)| \leq \frac{c_2}{u^{2n}} e^{-C_2 T^2/u^2 + C_2' T} \tag{3.7}$$

for any $x \in M$ with $d(x,F) \geq \alpha$. (3.6) and (3.7) imply that for some $c, C > 0$,

$$|P_{u,T/u}(x,x)| \leq c\, e^{-C/u^2}. \tag{3.8}$$

Formula (3.1) follows from [4, equation (12.7)]:

$$|P_{u,T/u}(g^{-1}(\theta)x, x)| \leq |P_{u,T/u}(g^{-1}(\theta)x, g^{-1}(\theta)x)|^{\frac{1}{2}} |P_{u,T/u}(x,x)|^{\frac{1}{2}} \tag{3.9}$$

and from the S^1-invariance of $|P_{u,T/u}(x,x)|$. q.e.d.

Proposition 3.2 can also be proved using the method of [13]. The proof here is similar to that of [6, Theorem 3.11] except that, without the $\mathbb{Z}_2$ symmetry there, we do not get a vanishing result in Proposition 3.4 below.

Definition 3.3. Let $R_p(\theta)$ be the isotropy representation of $e^{\sqrt{-1}\theta} \in S^1$ on T_pM and let $Z = (z^1, \cdots, z^n)$ be the linear complex coordinates on T_pM such that the action of $R_p(\theta)$ is

$$R_p(\theta)(z^1, \cdots, z^n) = (e^{\sqrt{-1}\lambda_1^p \theta} z^1, \cdots, e^{\sqrt{-1}\lambda_n^p \theta} z^n). \tag{3.10}$$

For $T \geq 0$, set

$$\mathcal{B}^{2,p}_{T^2} = \frac{1}{2}\Delta^p + \frac{1}{2}T^2 \sum_{k=1}^n |\lambda_k^p|^2 |z^k|^2 + T \sum_{k=1}^n \sqrt{-1}\lambda_k^p (e^k e^{\bar{k}} - i_k i_{\bar{k}}) \tag{3.11}$$

and

$$\mathcal{C}^{2,p}_{T^2} = \frac{1}{2}\Delta^p + \frac{1}{2}T^2 \sum_{k=1}^n |\lambda_k^p|^2 |z^k|^2 - \frac{1}{2}T \sum_{k=1}^n \lambda_k^p ([e^k, i_k] + [e^{\bar{k}}, i_{\bar{k}}]), \tag{3.12}$$

where Δ^p is the (positive) flat Laplacian on T_pM.

It is easy to see that the $SU(2)$ group elements $S_a(\alpha)$ $(a = 1, 2, 3)$ act on $\Omega^{*,*}(T_pM)$ and that

$$S_2(-\tfrac{\pi}{2})^{-1} \mathcal{C}^{2,p}_{T^2} S_2(-\tfrac{\pi}{2}) = \mathcal{B}^{2,p}_{T^2}. \tag{3.13}$$

This can be used to recover [3, Theorem 1.6] from [14]. Moreover, if $Q^p_{T^2}$ is the heat kernel associated to $\exp(-\mathcal{C}^{2,p}_{T^2})$, then $S_2(-\frac{\pi}{2})^{-1} Q^p_{T^2} S_2(-\frac{\pi}{2})$ is that of $\exp(-\mathcal{B}^{2,p}_{T^2})$.

Proposition 3.4. *For $T > 0$, $\theta \in \mathbb{R}$,*

$$\begin{aligned}\lim_{u\to 0} \operatorname{Tr}_{\Omega^{0,k}(M,E)} &\exp[-u^2\Box_{Th/u^2} - (\theta + \sqrt{-1}T)L_V] \\ &= \sum_{p\in F} E_p(\theta+\sqrt{-1}T)\operatorname{Tr}_{\Omega^{0,k}(T_pM)}[R_p(\theta)\exp(-\mathcal{C}^{2,p}_{T^2})].\end{aligned} \tag{3.14}$$

Moreover, the limit is uniform in θ.

Proof. We recall the notation of [5, §11-12]. Fix a small $\epsilon > 0$. For $p \in F$, the ball $B^{T_pM}(0,\epsilon) \subset T_pM$ is identified with the ball $B^M(0,\epsilon) \subset M$ by the exponential map. Let $k'(Z) = \det(d_Z \exp)$, $Z \in T_pM$, be the Jacobian. Then $dv_{T_pM}(Z) = k'(Z)dv_M(Z)$ and $k(0) = 1$. We also identify T_ZM, E_Z with T_pM, E_p, respectively, by the parallel transports along the geodesic connecting p and Z. The operators D^M and $\hat{V}$ now act on smooth sections of $\Lambda^*(T_pM)\otimes E_p$ over $B^{T_pM}(0,\epsilon)$. The setup here is simpler than that in [5], [4] because F is discrete and we choose the resolution (3.2). (Using the notation in [5, §8.f], here $\xi^+ = 0$ and $\xi^- = \xi$.) Following [5, §11.h-i and §12.d-e], we define

$$L^{1,p}_{u,T} = u^2(1-\rho^2(Z))\frac{\Delta^p}{2} + \rho^2(Z)(u^2\Box_{Tv/u} - r_{u,T}), \tag{3.15}$$

where $\rho(Z) = \rho(|Z|)$ is a smooth function such that $\rho(Z) = 1$ if $|Z| \le \frac{\epsilon}{4}$ and $\rho(Z) = 0$ if $|Z| \ge \frac{\epsilon}{2}$, and

$$L^{3,p}_{u,T} = F_u^{-1}L^{1,p}_{u,T}F_u, \tag{3.16}$$

where F_u is a rescaling: $F_u h(Z) = h(Z/u)$. Let $P^{1,p}_{u,T}(Z,Z')$, $P^{3,p}_{u,T}(Z,Z')$ be the smooth heat kernel associated to the operators $\exp(-L^{1,p}_{u,T})$, $\exp(-L^{3,p}_{u,T})$, respectively, calculated in the volume element dv_{T_pM}. Clearly

$$u^{2n}P^{1,p}_{u,T}(uZ,uZ') = P^{3,p}_{u,T}(Z,Z'). \tag{3.17}$$

The only term in $L^{3,p}_{u,T}$ that did not appear in [5, equation (11.60)] is $-\rho^2(uZ)r_{u,T}(uZ)$. It is easy to see that for $u \in (0,1]$, $T \in [1,1/u]$, the operator $1_{u|Z|\le\epsilon/2}r_{u,T}(uZ)$ is uniformly bounded with respect to the norm $|\cdot|_{u,T,0,0}$ in [5, Definition 11.23]. This, together with [5, Proposition 11.24], is enough to establish the results in [5, Theorem 11.26] (in the special case of $Z_0 = 0$) for $L^{3,p}_{u,T}$. We can then proceed as the proof of [5, Theorem 11.31] and obtain the analog of [5, Theorem 12.14] on the uniform estimates of $P^{3,p}_{u,T/u}$. In particular, for any $m \in \mathbb{N}$, there exists $c > 0$ such that if $u \in (0,1]$, then

$$|P^{3,p}_{u,T/u}(Z,Z)| \le \frac{c}{(1+|Z|)^m} \tag{3.18}$$

for $|Z| \le \frac{\epsilon}{8u}$. Using (3.17) and the analog of (3.9), we get

$$u^{2n}|P^{1,p}_{u,T/u}(uR_p^{-1}(\theta)Z, uZ)| \le \frac{c}{(1+|Z|)^m}. \tag{3.19}$$

Next, from (3.15) and (3.16) it follows that

$$\begin{aligned} L^{3,p}_{u,T/u} =& \tfrac{1}{2}u^2(1-\rho^2(uZ))\Delta^p + \rho^2(uZ)(D^M)^2 \\ &+ \rho^2(uZ)(T\{D^M,\hat{V}\} + u^{-2}T^2\hat{V}^2(uZ) - r_{u,T/u}(uZ)). \end{aligned} \tag{3.20}$$

It is easy to see that as $u \to 0$, $r_{u,T/u}(uZ) \to \sqrt{-1}Tr_V(p)$; the rest of the terms in (3.20) tends to $\mathcal{B}^{2,p}_{T^2}$ by [5, Propositions 12.10, 12.12]. Hence

$$L^{3,p}_{u,T/u} \to \mathcal{B}^{2,p}_{T^2} - \sqrt{-1}Tr_V(p), \quad \text{as } u \to 0. \tag{3.21}$$

Proceed as the proof of [5, Theorem 12.16] (with the simplification $L_{u,1} = L^{3,p}_{u,T/u}$ and $L_{u,2}, L_{u,3}, L_{u,4} = 0$) and as [5, §12.i], we conclude that

$$\begin{aligned} &S_2(-\tfrac{\pi}{2})^{-1}P^{3,p}_{u,T/u}S_2(-\tfrac{\pi}{2}) \\ &\quad \to S_2(-\tfrac{\pi}{2})^{-1}Q^p_{T^2}S_2(-\tfrac{\pi}{2}) \otimes e^{\sqrt{-1}Tr_V(p)}, \quad \text{as } u \to 0 \end{aligned} \tag{3.22}$$

in the sense of distributions on $T_pM \times T_pM$. By the uniform estimates on $P^{3,p}_{u,T/u}$,

$$\begin{aligned} &e^{-\theta L_V}P^{3,p}_{u,T/u}(R^{-1}(\theta)Z,Z) \\ &\quad \to R_p(\theta)Q^p_{T^2}(R_p^{-1}(\theta)Z,Z) \otimes e^{(\theta+\sqrt{-1}T)r_V(p)}, \quad \text{as } \ u \to 0 \end{aligned} \tag{3.23}$$

uniformly in θ and in Z belonging to any compact set in T_pM. Using (3.17) and taking the (local) trace over anti-holomorphic forms only, we get

$$\begin{aligned} &\lim_{u\to 0} u^{2n}\,\mathrm{tr}_{\Omega^{0,k}_p \otimes E_p}[e^{-\theta L_V}P^{1,p}_{u,T/u}(uR^{-1}(\theta)Z,uZ)] \\ &\quad = E_p(\theta+\sqrt{-1}T)\,\mathrm{tr}_{\Omega^{0,k}_p}[R_p(\theta)Q^p_{T^2}(R_p^{-1}(\theta)Z,Z)]. \end{aligned} \tag{3.24}$$

The arguments leading to (3.6) imply (see [5, §12.d] and [4, §12.d]) that there are $c_0, C_0 > 0$ such that for all $u \in (0,1]$ and $Z \in T_pM$ with $|Z| \leq \frac{\epsilon}{8}$, we have

$$\begin{aligned} |P_{u,T/u}((p, R_p^{-1}(\theta)Z), (p, Z))k'(Z) - P^{1,p}_{u,T/u}(R_p^{-1}(\theta)Z, Z)| \\ \leq c_0 e^{-C_0/u^2}. \end{aligned} \tag{3.25}$$

Therefore in (3.24) and (3.19), $P^{1,p}_{u,T/u}(uR_p^{-1}(\theta)Z, uZ)$ can be replaced by $P_{u,T/u}((p, uR_p^{-1}(\theta)Z), (p, uZ))k'(uZ)$ for $|Z| \leq \frac{\epsilon}{8u}$. Thus the dominated convergence theorem (as in [5, Remark 12.5], but adapted to take into account uniform convergence) yields

$$\begin{aligned} &\lim_{u \to 0} \int_{B^M(F,\epsilon/8)} \mathrm{tr}_{\Omega^{0,k}_x \otimes E_x}[e^{-\theta L_V} P_{u,T/u}(g^{-1}(\theta)x, x)] dv_M(x) \\ &\quad = \sum_{p \in F} E_p(\theta + \sqrt{-1}T) \\ &\qquad \cdot \int_{T_pM} \mathrm{tr}_{\Omega^{0,k}_p}[R_p(\theta) Q^p_{T^2}(R_p^{-1}(\theta)Z, Z)] dv_{T_pM}(Z) \end{aligned} \tag{3.26}$$

uniformly in θ. By Proposition 3.2, we can replace the domain of the integration on the left-hand side by M and thus the proposition follows.

q.e.d.

Definition 3.5. Let $q(u,\theta) = \sum_{m \in \mathbb{Z}} q_m(u) e^{\sqrt{-1}m\theta}$ be a family of formal characters of S^1 parameterized by $u \in \mathbb{R}$ and let $q(\theta) = \sum_{m \in \mathbb{Z}} q_m e^{\sqrt{-1}m\theta} \in \mathbb{R}((e^{\sqrt{-1}\theta}))$. We say that $\lim_{u \to u_0} q(u,\theta) = q(\theta)$ in $\mathbb{R}((e^{\sqrt{-1}\theta}))$ if for all $m \in \mathbb{Z}$, $\lim_{u \to u_0} q_m(u) = q_m$.

Corollary 3.6. *For $T > 0$, the limit (3.14) holds in $\mathbb{R}((e^{\sqrt{-1}\theta}))$.*

Proof. By Proposition 3.4, we know that as $u \to 0$,

$$\begin{aligned} &\mathrm{Tr}_{\Omega^{0,k}(M,E)} \exp[-u^2 \Box_{Th/u^2} - (\theta + \sqrt{-1}T) L_V] \\ &\quad - \sum_{p \in F} E_p(\theta + \sqrt{-1}T)\, \mathrm{Tr}_{\Omega^{0,k}(T_pM)}[R_p(\theta) \exp(-\mathcal{C}^{2,p}_{T^2})] \to 0 \end{aligned} \tag{3.27}$$

uniformly in θ, and hence in $L^2(S^1)$ as well. This implies that all the Fourier coefficients of the left-hand side tend to 0. The result follows.

q.e.d.

4. Proof of the theorem

As explained in the introduction, the heat kernel proof of the equivariant Morse-type inequalities is based on the following.

Lemma 4.1. *For $u > 0$, $T > 0$, we have*

$$\begin{aligned}\sum_{k=0}^{n} t^k \operatorname{Tr}_{\Omega^{0,k}(M,E)} \exp(-u^2\overline{\Box}_{Th/u^2} - \theta L_V) \\ = \sum_{k=0}^{n} t^k H^k(\theta) + (1+t)Q_{u,T}(\theta,t)\end{aligned} \tag{4.1}$$

in $\mathbb{R}((e^{\sqrt{-1}\theta}))[t]$ *for some* $Q_{u,T}(\theta,t) \geq 0$.

Proof. Recall that $\overline{\Box}_{Th/u^2} = \{\bar\partial_{Th/u^2}, \bar\partial^*_{Th/u^2}\}$. Since $\bar\partial_{Th/u^2}$ and $\bar\partial$ differ by an S^1-invariant conjugation, their cohomologies are isomorphic as representations of S^1. Using the (S^1-equivariant) Hodge decomposition, we get

$$\begin{aligned}&\operatorname{Tr}_{\Omega^{0,k}(M,E)} \exp(-u^2\overline{\Box}_{Th/u^2} - \theta L_V) \\ &= H^k(\theta) + \operatorname{Tr}_{\bar\partial^*_{Th/u^2}\Omega^{0,k+1}(M,E)} \exp(-u^2\bar\partial^*_{Th/u^2}\bar\partial_{Th/u^2} - \theta L_V) \\ &\quad + \operatorname{Tr}_{\bar\partial_{Th/u^2}\Omega^{0,k-1}(M,E)} \exp(-u^2\bar\partial_{Th/u^2}\bar\partial^*_{Th/u^2} - \theta L_V)\end{aligned} \tag{4.2}$$

as formal characters of S^1. Notice that the spectrum of the operator $\bar\partial^*_{Th/u^2}\bar\partial_{Th/u^2}$ on (the closure of) $\bar\partial^*_{Th/u^2}\Omega^{0,k+1}(M,E)$ is identical to that of $\bar\partial_{Th/u^2}\bar\partial^*_{Th/u^2}$ on (the closure of) $\bar\partial_{Th/u^2}\Omega^{0,k-1}(M,E)$. Since the S^1-action commutes with all the operators, we obtain

$$\begin{aligned}&\operatorname{Tr}_{\bar\partial^*_{Th/u^2}\Omega^{0,k+1}(M,E)} \exp(-u^2\bar\partial^*_{Th/u^2}\bar\partial_{Th/u^2} - \theta L_V) \\ &= \operatorname{Tr}_{\bar\partial_{Th/u^2}\Omega^{0,k}(M,E)} \exp(-u^2\bar\partial_{Th/u^2}\bar\partial^*_{Th/u^2} - \theta L_V) \geq 0\end{aligned} \tag{4.3}$$

in $\mathbb{R}((e^{\sqrt{-1}\theta}))$. We denote either of the expressions in (4.3) by $Q^k_{u,T}(\theta)$. Summing over $k = 0, \cdots, n$ in (4.2) gives (4.1) with

$$Q_{u,T}(\theta,t) = \sum_{k=0}^{n} Q^k_{u,T}(\theta)t^k \geq 0.$$

q.e.d.

We now take the limit $u \to 0$. To use Proposition 3.4 or Corollary 3.6, we need the following result on the equivariant heat kernel of the anti-holomorphic sector of the (supersymmetric) harmonic oscillator.

Lemma 4.2. *For $T > 0$,*

$$\begin{aligned}&\mathrm{Tr}_{\Omega^{0,k}(T_pM)}[R_p(\theta)\exp(-\mathcal{C}^{2,p}_{T^2})]\\ &= \sum_{I\subset\{1,\cdots,n\},|I|=k} \frac{e^{-T\sum_{k=0}^{n}|\lambda_k^p| - T\sum_{k\notin I}\lambda_k^p + \sqrt{-1}\theta\sum_{k\in I}\lambda_k^p}}{\prod_{k=1}^{n}[(1-e^{-(T-\sqrt{-1}\theta)|\lambda_k^p|})(1-e^{-(T+\sqrt{-1}\theta)|\lambda_k^p|})]}.\end{aligned} \tag{4.4}$$

Proof. The operator $\mathcal{C}^{2,p}_{T^2}$ acting on $\Omega^{0,*}(T_pM)$ splits S^1-equivariantly to n copies of

$$\mathcal{C}^2_{T^2} = \tfrac{1}{2}\Delta + \tfrac{1}{2}T^2|\lambda|^2|z|^2 - \tfrac{1}{2}T\lambda(-1 + [d\bar{z}\wedge, i_{\partial/\partial\bar{z}}]), \quad \lambda \in \mathbb{Z}\backslash\{0\} \tag{4.5}$$

acting on $\Omega^{0,*}(\mathbb{C})$. Here S^1 acts on $\mathbb{C}$ by $R(\theta) = e^{\sqrt{-1}\lambda\theta}$ and hence on $\Omega^{0,*}(\mathbb{C})$ as well. (4.5) is the sum of the Hamiltonian for the two dimensional harmonic oscillator

$$\mathcal{H}^2_{T^2} = \tfrac{1}{2}\Delta + \tfrac{1}{2}T^2|\lambda|^2|z|^2 \tag{4.6}$$

and a bounded operator of order zero. The smooth heat kernel associated to the operator $\exp(-\mathcal{H}^2_{T^2})$ acting on $\Omega^0(\mathbb{C})$ is given by Mehler's formula

$$\begin{aligned}&K_{T^2}(z, z')\\ &= \frac{T|\lambda|}{2\pi\sinh T|\lambda|}\exp\left[-T|\lambda|\left(\frac{|z|^2+|z'|^2}{2\tanh T|\lambda|} - \frac{\mathrm{Re}(\bar{z}z')}{\sinh T|\lambda|}\right)\right].\end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned}&\mathrm{Tr}_{\Omega^0(\mathbb{C})}[R(\theta)\exp(-\mathcal{H}^2_{T^2})]\\ &= \int_{\mathbb{C}} d^2z\, K_{T^2}(e^{-\sqrt{-1}\lambda\theta}z, z)\\ &= \frac{e^{-T|\lambda|}}{(1-e^{-(T-\sqrt{-1}\theta)|\lambda|})(1-e^{-(T+\sqrt{-1}\theta)|\lambda|})}.\end{aligned} \tag{4.8}$$

The bounded 0-th order operator in (4.5) takes values $T\lambda$ and 0, respectively, on 0- and 1-forms. Furthermore, the S^1-action $R(\theta)$ picks up a phase $e^{\sqrt{-1}\lambda\theta}$ on $d\bar{z}$. Therefore

$$\begin{aligned}&\mathrm{Tr}_{\Omega^{0,k}(\mathbb{C})}[R(\theta)\exp(-\mathcal{C}^2_{T^2})]\\ &\quad=\begin{cases}\frac{e^{-T|\lambda|-T\lambda}}{(1-e^{-(T-\sqrt{-1}\theta)|\lambda|})(1-e^{-(T+\sqrt{-1}\theta)|\lambda|})}, & \text{if } k=0,\\ \frac{e^{-T|\lambda|+\sqrt{-1}\lambda\theta}}{(1-e^{-(T-\sqrt{-1}\theta)|\lambda|})(1-e^{-(T+\sqrt{-1}\theta)|\lambda|})}, & \text{if } k=1.\end{cases}\end{aligned}\tag{4.9}$$

Returning to the problem on T_pM, for $I=\{i_1,\cdots,i_k\}\subset\{1,\cdots,n\}$, set $d\bar{z}^I=d\bar{z}^{i_1}\wedge\cdots\wedge d\bar{z}^{i_k}$. Then we have

$$\begin{aligned}&\mathrm{Tr}_{\Omega^0(T_pM)d\bar{z}^I}[R_p(\theta)\exp(-\mathcal{C}^{2,p}_{T^2})]\\ &\quad=\frac{e^{-T\sum_{k=0}^n|\lambda_k^p|-T\sum_{k\notin I}\lambda_k^p+\sqrt{-1}\theta\sum_{k\in I}\lambda_k^p}}{\prod_{k=1}^n[(1-e^{-(T-\sqrt{-1}\theta)|\lambda_k^p|})(1-e^{-(T+\sqrt{-1}\theta)|\lambda_k^p|})]}.\end{aligned}\tag{4.10}$$

The trace on $\Omega^{0,k}(T_pM)$ is the sum of (4.10) over I with $|I|=k$. q.e.d.

Since $T>0$, the standard expansion of the right-hand side of (4.4) yields a convergent series. By the spectral theorem and the uniqueness of Fourier series, (4.4) can be interpreted as an equality of formal characters of S^1 according to the expansion in (1.9).

Proof of formula (1.11). In (4.1) we replace θ formally by $\theta+\sqrt{-1}T$ and still regard it as an equality of formal series in $e^{\sqrt{-1}\theta}$. Since as $u\to 0$ the limit of the left-hand side exists in $\mathbb{R}((e^{\sqrt{-1}\theta}))$ (Corollary 3.6) and $H^k(\theta)$ is independent of u, we conclude that $\lim_{u\to 0}Q_{u,T}(\theta+\sqrt{-1}T,t)=Q_T(\theta+\sqrt{-1}T,t)$ also exists and that $Q_T(\theta,t)\geq 0$. Therefore

$$\begin{aligned}&\sum_{p\in F}E_p(\theta+\sqrt{-1}T)\,\mathrm{Tr}_{\Omega^{0,k}(T_pM)}[R_p(\theta)\exp(-\mathcal{C}^2_{T^2})]\\ &\quad=\sum_{k=0}^n t^kH^k(\theta+\sqrt{-1}T)+(1+t)Q_T(\theta+\sqrt{-1}T,t).\end{aligned}\tag{4.11}$$

Using Lemma 4.2 and changing $\theta+\sqrt{-1}T$ back to θ, we get

$$\begin{aligned}&\sum_{p\in F,I\subset\{1,\cdots,n\}}t^{|I|}E_p(\theta)\frac{e^{-T\sum_{k=1}^n|\lambda_k^p|-T\sum_{k\notin I}\lambda_k^p+(T+\sqrt{-1}\theta)\sum_{k\in I}\lambda_k^p}}{\prod_{k=1}^n[(1-e^{\sqrt{-1}|\lambda_k^p|\theta})(1-e^{-|\lambda_k^p|(2T+\sqrt{-1}\theta)})]}\\ &\quad=\sum_{k=0}^n t^kH^k(\theta)+(1+t)Q_T(\theta,t).\end{aligned}\tag{4.12}$$

Finally, as $T \to +\infty$, the limit of each summand on the left-hand side is 0 except when the pair (p, I) satisfies $I = \{k \mid \lambda_k^p > 0\}$, in which case the limit is

$$\frac{t^{n-n_p} E_p(\theta) e^{\sqrt{-1}\sum_{\lambda_k^p>0}|\lambda_k^p|\theta}}{\prod_{k=1}^{n}(1 - e^{\sqrt{-1}|\lambda_k^p|\theta})}.$$

Consequently, $\lim_{T\to+\infty} Q_T(\theta, t) = Q^-(\theta, t)$ exists as well, and $Q^-(\theta, t) \geq 0$. Hence formula (1.11) is proved. q.e.d.

Remark 4.3. As pointed out by the referee, the left-hand side of (4.1) is analytic in $\theta \in \mathbb{C}$ since the operator $-u^2\overline{\Box}_{Th/u^2} - \theta L_V$ is a suitable perturbation of the Laplacian [11, Ch. 9, Theorem 2.6]. Propositions 3.2 and 3.4 show that when $\mathrm{Im}\,\theta = T$, the heat kernel of the above operator localizes to the fixed-point set F as $u \to 0$ and the limit is uniform in $\mathrm{Re}\,\theta$. It seems that these results remain valid when $0 < \mathrm{Im}\,\theta < 2T$.

Acknowledgement

We are grateful to Weiping Zhang for bringing to our attention and kindly explaining the works in [2]–[6], and for critical comments on the manuscript. We also thank Sixia Yu for discussions. S. Wu thanks the hospitality of the University of Adelaide, where this work started. Finally, we thank the referee for many constructive suggestions.

References

[1] M. F. Atiyah & R. Bott, *A Lefschetz fixed point formula for elliptic complexes,* Part I, Ann. of Math. **86** (1967) 374–407; Part II, Ann. of Math. **87** (1968) 451–491.

[2] J.-M. Bismut, *The Witten complex and the degenerate Morse inequalities*, J. Differential Geom. **23** (1986) 207–240.

[3] ———, *Koszul complexes, harmonic oscillators, and the Todd class*, J. Amer. Math. Soc. **3** (1990) 159–256.

[4] ———, *Equivariant immersions and Quillen metrics*, J. Differential Geom. **41** (1995) 53–157.

[5] J.-M. Bismut & G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. **74** (1991) 1–297.

[6] J.-M. Bismut & W. Zhang, *Real embeddings and eta invariants*, Math. Ann. **295** (1993) 661–684.

[7] J. P. Demailly, *Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne*, Séminaire d'analyse, Lecture Notes in Math., Vol. 1198 (ed. P. Lelong et. al.), Springer, Berlin, 1986, 88–97.

[8] ———, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*, Ann. Inst. Fourier (Grenoble), **35** (1985) 185–229; *Holomorphic Morse inequalities*, Several complex variables and complex geometry, Part 2, (Proc. Sympos. Pure Math., Vol. 52), eds. E. Bedford et. al. Amer. Math. Soc., 1991, 93–114.

[9] T. Frankel, *Fixed points and torsion on Kähler manifold*, Ann. of Math. **70** (1959) 1–8.

[10] P. Griffiths & J. Harris, *Principles of algebraic geometry*, John Wiley, New York, 1978, §0.7.

[11] T. Kato, *Perturbation theory for linear operators,* 2nd ed., Springer, Berlin, 1995.

[12] K. Liu, *Holomorphic equivariant cohomology*, Math. Ann. **303** (1995) 125–148.

[13] J. Roe, *Elliptic operators, topology and asymptotic methods,* Pitman Res. Notes Math. Ser. 179, 1988, §12.

[14] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17**(1982) 661–692.

[15] ———, *Holomorphic Morse inequalities*, Algebraic and differential topology, Teubner-Texte Math. **70** (ed. G. Rassias), Teubner, Leipzig, 1984, 318–333.

[16] S. Wu, *Equivariant holomorphic Morse inequalities II: torus and non-Abelian group actions*, Math. Sci. Res. Inst., preprint No. 1996-013, `dg-ga/9602008`, 1996.

UNIVERSITY OF ADELAIDE, SOUTH AUSTRALIA
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, TRIESTE AND
MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY